GLOBAL EXISTENCE RESULTS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH QUADRATIC POTENTIALS

RÉMI CARLES

Abstract. We prove that no finite time blow up can occur for nonlinear Schrödinger equations with quadratic potentials, provided that the potential has a sufficiently strong repulsive component. This is not obvious in general, since the energy associated to the linear equation is not positive. The proof relies essentially on two arguments: global in time Strichartz estimates, and a refined analysis of the linear equation, which makes it possible to use continuity arguments and to control the nonlinear effects.

1. Introduction

We consider the nonlinear Schrödinger equation on $\mathbb{R}^n$,

$$i\partial_t u + \frac{1}{2}\Delta u = V(x)u + \lambda |u|^{2\sigma} u \quad ; \quad u\big|_{t=0} = u_0\,, \tag{1.1}$$

when the potential V is quadratic in x, or more generally when V is a second order polynomial. The general assumptions we make are the following: the space variable x is in $\mathbb{R}^n$, $\lambda \in \mathbb{R}$, $\sigma > 0$ with $\sigma < \frac{2}{n-2}$ if $n \geq 3$ (the nonlinearity is H^1–sub-critical), and

$$u_0 \in \Sigma := \left\{ f \in \mathcal{S}'(\mathbb{R}^n)\ ;\ \|f\|_\Sigma := \|f\|_{L^2(\mathbb{R}^n)} + \|\nabla f\|_{L^2(\mathbb{R}^n)} + \|xf\|_{L^2(\mathbb{R}^n)} < +\infty \right\}.$$

This space is very natural when one studies the case of the harmonic potential $V(x) \equiv |x|^2$, see e.g. [8]. When the nonlinearity is L^2–sub-critical ($\sigma < 2/n$), one can even consider initial data in L^2 only. When $V \in C^\infty(\mathbb{R}^n;\mathbb{R})$ is sub-quadratic ($\partial^\alpha V \in L^\infty(\mathbb{R}^n)$ for $|\alpha| \geq 2$), it is easy to prove existence and uniqueness of solutions to (1.1) in Σ locally in time, without making any assumption on the sign of V (see e.g. [8]). This follows for instance from the fact that dispersive estimates for the linear equation ($\lambda = 0$) are available for small time intervals (using perturbation arguments, one can construct a parametrix, see [14, 15]). One can then apply a fixed point argument on the Duhamel's formula associated to (1.1), on a closed system involving u, $\nabla_x u$ and xu.

When V is exactly the above isotropic harmonic potential, it is easy to prove global existence of the solution of (1.1) in Σ, when $\lambda > 0$ for instance, thanks to the conservations of mass and energy:

$$\begin{aligned} &\|u(t)\|_{L^2} \equiv \|u_0\|_{L^2}\ , \\ &E_V := \frac{1}{2}\|\nabla_x u(t)\|_{L^2}^2 + \frac{\lambda}{\sigma+1}\|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \int_{\mathbb{R}^n} V(x)\,|u(t,x)|^2\,dx = \text{const.} \end{aligned} \tag{1.2}$$

When V is nonnegative (or just bounded from below), and $\lambda > 0$, these conservations yield *a priori* estimates from which global existence follows.

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The opposite case is when $V(x) = -|x|^2$ is the repulsive harmonic potential. The quadratic case is critical on the one hand to ensure that the operator $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$: for $-\Delta - |x|^4$, this property fails (classical trajectories can reach infinite speed, see [21, 12]). On the other hand, it was proved in [5] that despite this first negative impression, the repulsive harmonic potential tends to encourage global existence. Of course, such results cannot be proved from the single conservations (1.2), since the linear energy ($\lambda = 0$) is not even a positive functional. Global existence in the case $\lambda > 0$ for instance stems from a conservation law, which can be viewed as the analog of the pseudo-conformal conservation law of the nonlinear Schrödinger equation with no potential (see [5] for more details).

The above mentioned law seems to hold only for *isotropic* repulsive harmonic potentials. If $n \geq 2$ and $V(x) = -x_1^2$, then no global existence result seems to be available, even in the case $\lambda > 0$. The aim of this paper is to give sufficient conditions on the potential V so that the solution u is global in time. This issue is part of the more general framework to understand the interaction between the linear dynamics generated by $-\Delta + V$, and nonlinear effects.

As was already exploited in different contexts [3, 5, 6], the fundamental solution for the linear problem is given explicitly when the potential V is a second order polynomial. This "miracle" is known as *Mehler's formula* (see [13, 17]), whose expression is given below (see (2.4)), and which is closely related to the fact that for such potentials, everything is known about classical trajectories. On the other hand, there is a gap between this nice framework and a more general case (see [6] for a more quantified discussion). This is why our study is restricted to potentials which are second order polynomials.

After reduction (see [6]), we may suppose that

$$V(x) = \sum_{j=1}^{n} \left(\delta_j \frac{\omega_j^2}{2} x_j^2 + b_j x_j \right) \ , \ n \geq 1 \, ,$$

where $\omega_j > 0$, $\delta_j \in \{-1, 0, 1\}$ and $\delta_j b_j = 0$ for any j. As noticed in [7], the Avron–Herbst formula makes it possible to remove the linear terms without affecting the global existence issue: we assume $b_j = 0$ for any j, and V is of the form

$$V(x) = \sum_{j=1}^{n} \delta_j \frac{\omega_j^2}{2} x_j^2 \ ; \ n \geq 1, \ \omega_j > 0, \ \delta_j \in \{-1, 0, 1\} \, , \quad \delta_1 = -1 \, . \tag{1.3}$$

The last assumption means that we do not consider positive potentials, for which several results are available (see e.g. [8]). We denote

$$H_V = -\frac{1}{2}\Delta + V \quad ; \quad U_V(t) = e^{-itH_V} \, .$$

Even though the energy E_V defined in (1.2) is not a positive functional in the linear case ($\lambda = 0$) when V satisfies (1.3), we can prove global existence in the L^2 sub-critical case $\sigma < \frac{2}{n}$, thanks to a careful analysis of Strichartz estimates:

Proposition 1.1. *Let $n \geq 1$, $\lambda \in \mathbb{R}$, $\sigma < \frac{2}{n-2}$ if $n \geq 3$, and V satisfying* (1.3)*.*
1. *Suppose $u_0 \in L^2(\mathbb{R}^n)$.*
   (i) *If $\sigma < \frac{2}{n}$, then* (1.1) *has a unique solution*
$$u \in C \cap L^\infty \left(\mathbb{R}; L^2(\mathbb{R}^n)\right) \cap L^q_{\mathrm{loc}} \left(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^n)\right) \, ,$$
   where $q = \frac{4\sigma+4}{n\sigma}$. In addition, the L^2-norm of $u(t, \cdot)$ is independent of time.

(ii) *If $\sigma \leq \frac{2}{n}$, then there exists $\delta = \delta(\sigma, n, |\lambda|, V) > 0$ such that if $\|u_0\|_{L^2} < \delta$, then* (1.1) *has a unique global solution $u \in C(\mathbb{R}; L^2) \cap L^{2+2\sigma}(\mathbb{R} \times \mathbb{R}^n)$. In addition, the L^2-norm of $u(t, \cdot)$ is independent of time, and there is scattering: there exist unique $u_-, u_+ \in L^2$ such that*

$$\|U_V(-t)u(t) - u_\pm\|_{L^2} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

2. *Suppose $u_0 \in \Sigma$.*

(i) *If $\sigma < \frac{2}{n}$, then the solution $u \in L^\infty\left(\mathbb{R}; L^2(\mathbb{R}^n)\right)$ of* (1.1) *is also in $C(\mathbb{R}; \Sigma)$, and satisfies* (1.2).

(ii) *There exists $\eta = \eta(\sigma, n, |\lambda|, V) > 0$ such that if $\|u_0\|_\Sigma < \eta$, then* (1.1) *has a unique global solution $u \in C(\mathbb{R}; \Sigma)$. In addition, there is scattering: there exist unique $u_-, u_+ \in \Sigma$ such that*

$$\|U_V(-t)u(t) - u_\pm\|_{\Sigma} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

Remark 1.2. When $\sigma < 2/n$, the points 1.i and 2.i hold more generally when V is a sub-quadratic potential (see Section 3).

For $\sigma \geq \frac{2}{n}$ and data not necessarily small, the basic example which we treat is when V is a horse shoe:

$$i\partial_t u + \frac{1}{2}\Delta u = \frac{1}{2}\left(-\omega_1^2 x_1^2 + \omega_2^2 x_2^2\right)u + \lambda|u|^{2\sigma}u \quad ; \quad u\big|_{t=0} = u_0\,. \tag{1.4}$$

In Section 4, we prove the following result:

Theorem 1.3. *Let $n \geq 2$, $\lambda \in \mathbb{R}$, $\sigma \geq \frac{2}{n}$ with $\sigma < \frac{2}{n-2}$ if $n \geq 3$, and $u_0 \in \Sigma$. Then there exists $\Lambda = \Lambda(n, \sigma, |\lambda|, \|u_0\|_\Sigma)$ such that for*

$$\omega_1 \geq \Lambda(1+\omega_2) + \frac{2\sigma^2}{2-(n-2)\sigma}(1+\omega_2)\ln(1+\omega_2)\,, \tag{1.5}$$

the solution u to (1.4) *is global in time: $u \in C(\mathbb{R}; \Sigma)$. Moreover, there is scattering: there exist unique $u_-, u_+ \in \Sigma$ such that*

$$\|U_V(-t)u(t) - u_\pm\|_{\Sigma} \underset{t\to\pm\infty}{\longrightarrow} 0\,.$$

The theorem has a heuristic explanation. The ω_1 term corresponds to a repulsive force: the larger ω_1, the stronger the repulsive force. On the other hand, the ω_2 term tends to confine the solution. There is a competition between these two effects. The above statement means that if the repulsive force dominates the confining one, then the solution cannot blow up in finite time, provided that in addition, $\omega_1 \gg 1$. The latter assumption is in the same spirit as the results of [5] in the case $\lambda < 0$ and $\sigma \geq 2/n$. The second part of the theorem means that nonlinear effects become negligible for large times. Theorem 1.3 yields a *sufficient* condition for global existence: it is not clear whether the assumption (1.5) is sharp or not (see Section 5 for a discussion).

Theorem 1.3 has a straightforward generalization, with a similar proof which we omit, because it involves heavier notations and bears no new difficulty.

Theorem 1.4. *Let $n \geq 2$, $\lambda \in \mathbb{R}$, $\sigma \geq \frac{2}{n}$ with $\sigma < \frac{2}{n-2}$ if $n \geq 3$, and $u_0 \in \Sigma$. Let V be of the form* (1.3), *and denote*

$$\omega_\pm = \max\{\omega_j\ ;\ \delta_j = \pm 1\} \quad (\omega_+ = 0 \textit{ if there is no } \delta_j = +1)\,.$$

Then there exists $\Lambda = \Lambda(n, \sigma, |\lambda|, \|u_0\|_\Sigma)$ *such that for*

$$\omega_- \geq \Lambda(1+\omega_+) + \frac{2\sigma^2}{2-(n-2)\sigma}(1+\omega_+)\ln(1+\omega_+)\,,$$

the solution u *to* (1.1) *is global in time,* $u \in C(\mathbb{R};\Sigma)$*, and there is scattering.*

As above, the statement can be summarized as follows: if the repulsive force is sufficiently strong compared to other effects (linear confinement is overcome if $\omega_- \gg \omega_+$, nonlinear effects are overcome if $\omega_- \gg 1$), then the solution is global and the nonlinearity can be viewed as a (short range) perturbation for large times.

The paper is organized as follows. In Section 2, we recall Strichartz estimates, and notice that they hold globally in time for U_V, when V is of the form (1.3). This follows from a simple remark after the proof of non-endpoint estimates in [19]. We deduce Proposition 1.1 in Section 3, and Theorem 1.3 in Section 4. In Section 5, we finally discuss the above results further into details.

2. Strichartz estimates

As recalled in the introduction, Strichartz estimates are the modern tool to study (among others) Schrödinger equations. More precisely, assume that U is a L^2*-unitary group*[1] such that:

$$\|U(t)\|_{L^1\to L^\infty} \lesssim |t|^{-\gamma}\,. \tag{2.1}$$

In the case of Schrödinger equations, we have $\gamma = \frac{n}{2}$, but it does not cost much to consider a general γ. Following [19], we say that a pair (q,r) is *sharp* γ*-admissible* if $q, r \geq 2$, $(q,r,\gamma) \neq (2,\infty,1)$ and

$$\frac{1}{q} + \frac{\gamma}{r} = \frac{\gamma}{2}\cdot$$

The following result is proved in [19] (see references therein for earlier proofs). For any sharp γ-admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$, there exist C_q and $C_{q,\widetilde{q}}$ such that for any time interval $I \ni 0$,

$$\|U(t)f\|_{L^q(I;L^r(\mathbb{R}^n))} \leq C_q \|f\|_{L^2(\mathbb{R}^n)}\ , \tag{2.2}$$

$$\left\|\int_0^t U(t-s)F(s)ds\right\|_{L^q(I;L^r(\mathbb{R}^n))} \leq C_{q,\widetilde{q}} \|F\|_{L^{\widetilde{q}'}(I;L^{\widetilde{r}'}(\mathbb{R}^n))}\ , \tag{2.3}$$

where a' stands for the Hölder conjugate exponent of a.

In order to explain the existence of this section, recall Mehler's formula ([13, 17]). If V is of the form (1.3), then denoting $H_V = -\frac{1}{2}\Delta + V$, we have:

$$U_V(t)f := e^{-itH_V}f = \prod_{j=1}^n \left(\frac{1}{2i\pi g_j(t)}\right)^{1/2}\int_{\mathbb{R}^n} e^{iS(t,x,y)}f(y)dy\,, \tag{2.4}$$

where

$$S(t,x,y) = \sum_{j=1}^n \frac{1}{g_j(t)}\left(\frac{x_j^2+y_j^2}{2}h_j(t) - x_jy_j\right),$$

[1] With Schrödinger equations with time-independent potentials in mind, this is natural.

and the functions g_j and h_j, related to the classical trajectories, are given by:

$$(g_j(t),h_j(t)) = \begin{cases} \left(\dfrac{\sinh(\omega_j t)}{\omega_j}, \cosh(\omega_j t)\right) \ , \ \text{if } \delta_j = -1\,, \\ (t,1) \ , \ \text{if } \delta_j = 0\,, \\ \left(\dfrac{\sin(\omega_j t)}{\omega_j}, \cos(\omega_j t)\right) \ , \ \text{if } \delta_j = +1\,. \end{cases} \tag{2.5}$$

Recall that if there exists $\delta_j = +1$, then e^{-itH_V} has some singularities, periodically in time (see e.g. [18]). This affects the above formula with phase factors we did not write (which can be incorporated in the definition of $(ig_j(t))^{1/2}$). However, these singularities may prevent the existence of *global in time* Strichartz estimates. Indeed, we obviously have

$$\|U_V(t)\|_{L^1\to L^\infty} \lesssim |t|^{-n/2} \quad \text{for } |t| \le \delta\,.$$

This yields local in time Strichartz estimates: the above constants C_q and $C_{q,\widetilde{q}}$ depend on the time interval I, and they may blow up on unbounded time intervals. This is obvious in the case of the isotropic harmonic potential, since eigenfunctions yield non-dispersive solutions (this can also be read from Mehler's formula). On the other hand, the exponential decay provided by a repulsive component of the potential (that is, there exists at least one $\delta_j = -1$) suggests that the singularities of confining forces may be balanced. We show that this is the case.

To see this, we simply remark that the above Strichartz estimates are still valid if we replace the assumption (2.1) by

$$\|U(t)\|_{L^1\to L^\infty} \le \mathtt{w}(t)^\gamma\,, \quad \text{with } \mathtt{w}\ge 0 \text{ and } \mathtt{w}\in L^1_w(\mathbb{R})\,, \tag{2.6}$$

the weak L^1 space[2]. This is straightforward, since the proof in [19] is actually valid with this relaxed assumption, for non-endpoint estimates. For the convenience of the reader, we recall the argument, with $I=\mathbb{R}$.

First, by duality, (2.2) is equivalent to:

$$\left\| \int U(-s)F(s)ds \right\|_{L^2(\mathbb{R}^n)} \lesssim \|F\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^n))}\ . \tag{2.7}$$

By the TT^* method, this is equivalent to

$$\left| \iint \langle U(-s)F(s), U(-t)G(t)\rangle ds dt \right| \lesssim \|F\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^n))} \|G\|_{L^{q'}(\mathbb{R};L^{r'}(\mathbb{R}^n))}\ .$$

By symmetry, it suffices to prove

$$|T(F,G)| \lesssim \|F\|_{L^{q'}_t L^{r'}_x} \|G\|_{L^{q'}_t L^{r'}_x}\ , \tag{2.8}$$

where $T(F,G) = \displaystyle\iint_{s<t} \langle U(-s)F(s), U(-t)G(t)\rangle ds dt$. Since U is unitary on L^2,

$$|\langle U(-s)F(s), U(-t)G(t)\rangle| \le \|F(s)\|_{L^2_x} \|G(s)\|_{L^2_x}\ .$$

We infer from the dispersive estimate (2.6) that

$$\begin{aligned} |\langle U(-s)F(s), U(-t)G(t)\rangle| &= |\langle U(t-s)F(s), G(t)\rangle| \\ &\le \mathtt{w}(t-s)^\gamma \|F(s)\|_{L^1_x} \|G(s)\|_{L^1_x}\ . \end{aligned}$$

[2]The assumption (2.1) is $\mathtt{w}(t) = c|t|^{-1}$ which is the basic example for a function in $L^1_w(\mathbb{R})$.

By interpolation,

$$|\langle U(-s)F(s),U(-t)G(t)\rangle| \le \mathtt{w}(t-s)^{2/q}\,\|F(s)\|_{L^{r'}_x}\,\|G(s)\|_{L^{r'}_x}\ ,$$

since (q,r) is sharp γ-admissible (and is not an endpoint). Then (2.8) follows from Hardy–Littlewood–Sobolev inequality (see e.g. [23] or [20, Sect. 4.3]). This proves the homogeneous estimate (2.2).

For the inhomogeneous case, consider two sharp γ-admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$. By duality, (2.3) is equivalent to:

$$|T(F,G)| \lesssim \|F\|_{L^{q'}_tL^{r'}_x}\,\|G\|_{L^{\widetilde{q}'}_tL^{\widetilde{r}'}_x}\ . \tag{2.9}$$

We have

$$|T(F,G)| \le \left(\sup_{t\in\mathbb{R}}\left\|\int_{s<t}U(-s)F(s)ds\right\|_{L^2_x}\right)\|G\|_{L^1_tL^2_x}\ ,$$

and when $(\widetilde{q},\widetilde{r})=(\infty,2)$, (2.9) follows from (2.7). Similarly, (2.9) holds when $(q,r)=(\infty,2)$. From (2.8), one has (2.9) when $(q,r)=(\widetilde{q},\widetilde{r})$. The general case (2.3) follows by interpolation between these three cases. We have precisely:

Lemma 2.1. *Let $(U(t))_{t\in\mathbb{R}}$ be a unitary group on $L^2(\mathbb{R}^n)$, satisfying the dispersive estimate* (2.6). *Then for any $T\in\overline{\mathbb{R}}_+$, and any sharp γ-admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$, we have*

$$\|U(t)f\|_{L^q(]-T,T[;L^r)} \le c_q\left\|\mathtt{w}\mathbb{1}_{]-2T,2T[}\right\|^{1/q}_{L^1_w}\|f\|_{L^2}\ ,$$

$$\left\|\int_0^t U(t-s)F(s)ds\right\|_{L^q(]-T,T[;L^r)} \le C_{q,\widetilde{q}}\left\|\mathtt{w}\mathbb{1}_{]-2T,2T[}\right\|^{1/q+1/\widetilde{q}}_{L^1_w}\|F\|_{L^{\widetilde{q}'}(]-T,T[;L^{\widetilde{r}'})}\ .$$

From now on, we shall simply write that a pair (q,r) is admissible when it is sharp $\frac{n}{2}$-admissible.

In the case of a quadratic potential (1.3), this yields global in time Strichartz estimates, since $\delta_1=-1$. From Mehler's formula (2.4), we have, for some $\delta>0$,

$$\|U_V(t)\|_{L^1\to L^\infty} \lesssim |t|^{-n/2}\quad \text{for } |t|\le\delta\,.$$

Now for $|t|>\delta$, the "worst" possible case is when, say, $\delta_1=-1$ and $\delta_j=+1$ for $j\ge 2$. Then

$$\|U_V(t)\|_{L^1\to L^\infty} \lesssim \left(e^{-\omega_1|t|}\prod_{j=2}^n\frac{1}{|\sin(\omega_jt)|}\right)^{1/2}\qquad \text{for } |t|>\delta\,.$$

Therefore, (2.6) is satisfied with

$$\mathtt{w}(t) = \text{const.}\left(\frac{1}{|t|}\mathbb{1}_{|t|\le\delta}+\left(e^{-\omega_1|t|}\prod_{j=2}^n\frac{1}{|\sin(\omega_jt)|}\right)^{1/n}\mathbb{1}_{|t|>\delta}\right)\,.$$

The first part is obviously in $L^1_w(\mathbb{R})$. The second part is in $L^1(\mathbb{R})$; this follows from Hölder's inequality, and the fact that

$$t\mapsto\left(\frac{e^{-|t|}}{|\sin t|}\right)^{1/n}\in L^{n-1}(\mathbb{R})\,.$$

Therefore, $\mathtt{w} \in L^1_w(\mathbb{R})$, and we have global in time Strichartz inequalities. In the case of (1.4), that is when $V(x) = \frac{1}{2}(-\omega_1^2 x_1^2 + \omega_2^2 x_2^2)$, we have:

$$\left\| \mathtt{w} \mathbb{1}_{]-\frac{\pi}{2\omega_2}, \frac{\pi}{2\omega_2}[} \right\|_{L^1_w} \lesssim 1 \quad \text{(always)}\,, \tag{2.10}$$

$$\|\mathtt{w}\|_{L^1_w} \lesssim 1 + \left(\frac{\omega_1}{\omega_2}\right)^{\frac{1}{n(n-1)}} e^{-\frac{\omega_1}{\omega_2}\frac{\pi}{2n(n-1)}}\,, \text{ if } \omega_1 \geq \omega_2\,. \tag{2.11}$$

(2.10) is straightforward; (2.11) follows from Hölder's inequality, writing $\mathtt{w}$ as the product of $n-1$ terms, for $|t| > \frac{\pi}{2\omega_2}$,

$$\mathtt{w}(t) = \frac{1}{\sqrt{2\pi}} \left(\left(\frac{\omega_1}{|\sinh(\omega_1 t)|}\right)^{\frac{1}{n-1}} \frac{\omega_2}{|\sin(\omega_2 t)|} \right) \times \prod_{j=1}^{n-2} \left(\frac{1}{|t|^{1/n}} \left(\frac{\omega_1}{|\sinh(\omega_1 t)|}\right)^{\frac{1}{n-1}} \right).$$

Remark 2.2. For $d > 0$, define the dispersive rate $\mathtt{w}_d$ by $\mathtt{w}(t)^{n/2} = \mathtt{w}_d(t)^{d/2}$. It is easy to check that $\mathtt{w}_d \in L^1_w(\mathbb{R})$ provided that $d \geq n$ (and $\mathtt{w}_d \in L^1(|x| > 1)$). Therefore, Lemma 2.1 shows that Strichartz inequalities hold with the same admissible pairs as in "space dimension d" (even though d needs not be an integer).

Remark 2.3. Notice that we have global in time Strichartz estimates in cases where there exist trapped trajectories. Consider for instance (1.4) with $n = 2$ and $\omega_1 = \omega_2 = 1$. In general, classical trajectories solve $\ddot{x} + \nabla V(x) = 0$; this yields in the present case:

$$\begin{aligned} x_1(t) &= x_1(0)\cosh t + \xi_1(0)\sinh t = \frac{e^t}{2}\left(x_1(0) + \xi_1(0)\right) + \frac{e^{-t}}{2}\left(x_1(0) - \xi_1(0)\right)\,, \\ x_2(t) &= x_2(0)\cos t + \xi_2(0)\sin t\,. \end{aligned}$$

If $x_1(0) + \xi_1(0) = 0$, then the trajectory is trapped in the future, but not in the past in general. If we suppose in addition that $x_1(0) = \xi_1(0) = 0$, then we have a trajectory which is trapped in the past *and* in the future (and nontrivial if $\xi_2(0) \neq 0$). Compare with the results of [10, 11]; it is proved that smoothing effects occur in the future provided that the classical trajectories are not trapped in the past. However, the results of [10] include potentials which grow at most linearly in x, and [11] does not consider the case of potentials. On the other hand, smoothing effects yield another approach to prove Strichartz estimates (see e.g. [22, 1, 2]). In our case, there exist trajectories trapped in the past and in the future, but global in time Strichartz estimates are available. It seems that the link between classical trajectories and (global in time) Strichartz estimates remains to be clarified.

3. Proof of Proposition 1.1

In the previous section, we saw that when V is of the form (1.3), U_V satisfies global in time Strichartz estimates. The constants in these inequalities may depend on $\omega_1, \ldots, \omega_n$, but this is not important in view of Proposition 1.1. As a matter of fact, global in time Strichartz estimates are really needed only for the third point of the proposition.

The first part of Proposition 1.1 is straightforward: one can mimic the proof given in the case of the nonlinear Schrödinger equation with no potential ($V \equiv 0$ in (1.1), see [24, 9, 8]). We recall the main argument. Duhamel's formula writes

$$u(t) = U_V(t)u_0 - i\lambda \int_0^t U_V(t-s)\left(|u|^{2\sigma}u\right)(s)ds. \tag{3.1}$$

Define $F(u)(t)$ as the right hand side of (3.1). The idea is to use a fixed point argument in the space given in Proposition 1.1. Introduce the following Lebesgue exponents:

$$r = 2\sigma + 2 \quad ; \quad q = \frac{4\sigma+4}{n\sigma} \quad ; \quad k = \frac{2\sigma(2\sigma+2)}{2-(n-2)\sigma} \cdot \tag{3.2}$$

Then (q,r) is the admissible pair of the proposition, and

$$\frac{1}{r'} = \frac{2\sigma}{r} + \frac{1}{r} \quad ; \quad \frac{1}{q'} = \frac{2\sigma}{k} + \frac{1}{q} \cdot$$

The main remark to prove the first point of Proposition 1.1 is that if $\sigma < \frac{2}{n}$, we have $\frac{1}{q} < \frac{1}{k}$, and Hölder's inequality in time yields

$$\|u\|_{L^k(I;L^r)} \leq |I|^{1/k-1/q} \|u\|_{L^q(I;L^r)} = |I|^{\frac{(2-n\sigma)(\sigma+1)}{2\sigma(2\sigma+2)}} \|u\|_{L^q(I;L^r)} \, .$$

The positive power of $|I|$ yields contraction in $L^\infty L^2 \cap L^q L^r$ for small time intervals, and the conservation of the L^2 norm of the solution shows global existence at the L^2 level. This proves 1.i.

If $\sigma = \frac{2}{n}$, then $k = q$, and Lemma 2.1 yields

$$\|u\|_{L^{2+\frac{4}{n}}(I\times\mathbb{R}^n)} \leq C\|u_0\|_{L^2} + C\|u\|^{1+\frac{4}{n}}_{L^{2+\frac{4}{n}}(I\times\mathbb{R}^n)} \, ,$$

for some constant C *independent of the time interval* I. The idea is then to use a bootstrap argument, for $\|u_0\|_{L^2}$ sufficiently small (see [9, 8] for details). When $\sigma < \frac{2}{n}$, one can use the same approach, thanks to Remark 2.2. Define $d = \frac{2}{\sigma} > n$. Then the nonlinearity is L^2–critical in "space dimension d", and the proof in [9] can be applied. This completes the proof of the first part of Proposition 1.1. Note that in the small data case, we used the fact that we have global in time Strichartz estimates, due to the repulsive character of the potential, $\delta_1 = -1$ in (1.3).

To prove the second point of Proposition 1.1, we restrict to the case of (1.4). This gives all the arguments for the general case, and prepares the proof of Theorem 1.3.

Introduce the operators

$$J_1(t) = \omega_1 x_1 \sinh(\omega_1 t) + i\cosh(\omega_1 t)\partial_1 \quad ; \quad H_1(t) = x_1\cosh(\omega_1 t) + i\frac{\sinh(\omega_1 t)}{\omega_1}\partial_1 \, ,$$

$$J_2(t) = -\omega_2 x_2 \sin(\omega_2 t) + i\cos(\omega_2 t)\partial_2 \quad ; \quad H_2(t) = x_2\cos(\omega_2 t) + i\frac{\sin(\omega_2 t)}{\omega_2}\partial_2 \, .$$

We define $J = (J_k)_{1\leq k\leq n}$ and $H = (H_k)_{1\leq k\leq n}$, where if $n \geq 3$,

$$J_k(t) = i\partial_k \quad ; \quad H_k(t) = x_k + it\partial_k \quad \text{for } k \geq 3 \, .$$

We have the weighted Gagliardo–Nirenberg inequalities:

Lemma 3.1. *Let $2 \leq p < \frac{2n}{n-2}$. There exists C_p independent of $\omega_1, \omega_2 > 0$ such that for any $f \in \Sigma$,*

$$\|f\|_{L^p} \leq C_p \left(\frac{\|J_1(t)f\|_{L^2}}{\cosh(\omega_1 t)}\right)^{\frac{\delta(p)}{n}} \left(\frac{\|J_2(t)f\|_{L^2}}{|\cos(\omega_2 t)|}\right)^{\frac{\delta(p)}{n}} \|f\|_{L^2}^{1-\delta(p)} \prod_{j=3}^{n} \|\partial_j f\|_{L^2}^{\frac{\delta(p)}{n}} \, ,$$

where $\delta(p) = n\left(\frac{1}{2} - \frac{1}{p}\right)$. *We also have*

$$\|f\|_{L^p} \leq C_p \left(\frac{\|J_1(t)f\|_{L^2}}{\cosh(\omega_1 t)}\right)^{\frac{\delta(p)}{n}} \left(\omega_2 \frac{\|H_2(t)f\|_{L^2}}{|\sin(\omega_2 t)|}\right)^{\frac{\delta(p)}{n}} \|f\|_{L^2}^{1-\delta(p)} \prod_{j=3}^{n} \|\partial_j f\|_{L^2}^{\frac{\delta(p)}{n}},$$

and therefore

$$\|f\|_{L^p} \lesssim \|f\|_{L^2}^{1-\delta(p)} \left(\frac{\|J_1(t)f\|_{L^2}}{\cosh(\omega_1 t)} \left(\|J_2(t)f\|_{L^2} + \omega_2 \|H_2(t)f\|_{L^2}\right) \prod_{j=3}^{n} \|\partial_j f\|_{L^2}\right)^{\frac{\delta(p)}{n}}.$$

This lemma follows from the usual Gagliardo–Nirenberg inequalities and:

$$\begin{aligned}
J_1(t) &= i\cosh(\omega_1 t) e^{i\omega_1 \frac{x_1^2}{2}\tanh(\omega_1 t)} \partial_1 \left(e^{-i\omega_1 \frac{x_1^2}{2}\tanh(\omega_1 t)} \cdot\right), \\
J_2(t) &= i\cos(\omega_2 t) e^{-i\omega_2 \frac{x_2^2}{2}\tan(\omega_2 t)} \partial_2 \left(e^{i\omega_2 \frac{x_2^2}{2}\tan(\omega_2 t)} \cdot\right), \\
H_2(t) &= i\frac{\sin(\omega_2 t)}{\omega_2} e^{i\omega_2 \frac{x_2^2}{2}\cot(\omega_2 t)} \partial_2 \left(e^{-i\omega_2 \frac{x_2^2}{2}\cot(\omega_2 t)} \cdot\right).
\end{aligned} \tag{3.3}$$

Note that the Galilean operators H_k, $k \geq 3$, share the same property:

$$H_k(t) = it\, e^{i\frac{x_k^2}{2t}} \partial_k \left(e^{-i\frac{x_k^2}{2t}} \cdot\right), \quad \text{for } k \geq 3. \tag{3.4}$$

To prove global existence in Σ, it is sufficient to prove that $A(t)u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; L^2)$ for any $A \in \{Id, J, H\}$. This follows from the formula

$$\begin{pmatrix} J_j(t) \\ H_j(t) \end{pmatrix} = \begin{pmatrix} -\delta_j \omega_j^2 g_j(t) & h_j(t) \\ h_j(t) & g_j(t) \end{pmatrix} \begin{pmatrix} x_j \\ i\partial_j \end{pmatrix}, \quad \forall j \geq 1.$$

The operators we use share the same properties as those which are used for ∇_x in the case with no potential: they commute with the linear part of the equation (including the potential V, since they are Heisenberg derivatives – see [6] and the discussion therein); they yield Gagliardo–Nirenberg inequalities; they act like derivatives on nonlinearities of the form $G(|z|^2)z$, from (3.3) and (3.4).

Fix $A \in \{Id, J, H\}$. From the above arguments and Strichartz estimates,

$$\begin{aligned}
\|A(t)F(u)\|_{L^q(I;L^r)} &\lesssim \left\|A(t)\left(|u|^{2\sigma}u\right)\right\|_{L^{q'}(I;L^{r'})} \lesssim \left\||u|^{2\sigma}A(t)u\right\|_{L^{q'}(I;L^{r'})} \\
&\lesssim \|u\|_{L^k(I;L^r)}^{2\sigma} \|A(t)u\|_{L^q(I;L^r)}.
\end{aligned}$$

Since $u \in L^q_{\mathrm{loc}}(\mathbb{R}; L^r)$ and $\frac{1}{q} < \frac{1}{k}$, we have $Au \in L^q_{\mathrm{loc}}(\mathbb{R}; L^r)$. Using Strichartz inequalities once more, we infer that $Au \in L^\infty_{\mathrm{loc}}(\mathbb{R}; L^2)$.

In the small data case, one can also resume the usual proof for global existence in Σ. Scattering for any $\sigma > 0$ (in this small data case) follows the same way as global in time Strichartz estimates, since Lemma 3.1 yields an exponential decay for u. This completes the proof of Proposition 1.1.

Remark 3.2. As pointed out in Remark 1.2, some results for $\sigma < 2/n$ hold for any sub-quadratic potential V. If $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$ is such that $\partial^\alpha V \in L^\infty(\mathbb{R}^n)$ for any $|\alpha| \geq 2$, then the results of [14, 15] prove that the group U_V has the same dispersion has the free group U_0 for *small times*. One can then mimic the proof of [24] to infer global existence at the L^2 level. Global existence in Σ follows, considering the closed system of estimates for $\nabla_x u$ and xu. On the other hand, no scattering result (even

for small data) must be expected under these general assumptions; the case of the harmonic potential yields a counterexample (see [4] for a high-frequency analysis).

4. Proof of Theorem 1.3

First, in the same spirit as in [5], we analyze the local existence result, to bound from below the local existence time, in term of the parameters ω_1 and ω_2. Then, we notice that we obtain a time at which u is defined and small, and for which therefore the nonlinearity is not too strong. We consider the solution of the linear equation ((1.4) with $\lambda = 0$) that coincides with u at that time. A continuity argument shows that u cannot move away too much from this linear solution. Since the linear solution is global, so is u. The scattering is then an easy by-product.

The first step is:

Proposition 4.1. *Let $u_0 \in \Sigma$. There exists $\delta = \delta(n, \sigma, |\lambda|, \|u_0\|_\Sigma) > 0$ independent of $\omega_1, \omega_2 \geq 0$, and $\frac{\delta}{1+\omega_2} \leq t_0 < \frac{\pi}{8\omega_2}$ such that* (1.4) *has a unique solution*

$$u \in C\left(]-2t_0, 2t_0[; \Sigma\right) \cap L^q\left(]-2t_0, 2t_0[; W^{1,2\sigma+2}(\mathbb{R}^n)\right) ,$$

with q given in (3.2)*. In addition, the conservations* (1.2) *hold, and there exists C_0 independent of $\omega_1, \omega_2 \geq 0$ such that*

$$\sup_{|t| \leq t_0} \|A(t)u\|_{L^2} \leq C_0 , \quad \forall A \in \{Id, J, H\} .$$

Remark 4.2. A similar result could be proved with slightly more general data than $u_0 \in \Sigma$, following exactly the same lines. Namely, we could assume

$$u_0 \in X_{1,2} := \left\{ f \in H^1(\mathbb{R}^n) \ ; \ x_1 f, x_2 f \in L^2(\mathbb{R}^n) \right\} .$$

Proof. For $|t| < \pi/(4\omega_2)$, Lemma 2.1 and (2.10) yield Strichartz estimates with constants independent of ω_1 and ω_2, and the weights in the first inequality of Lemma 3.1 are bounded uniformly in ω_1 and ω_2.

We can then mimic the usual proof (see e.g. [8]), that consists in applying a fixed point theorem on (3.1), using similar arguments as in the previous section. We give the main lines of the proof, so that the bound from below on t_0 is clear. Let $R := \|u_0\|_\Sigma$. With the definition (3.2), denote

$$Y_r(I) := \left\{ u \in C(I; \Sigma); \ A(t)u \in L^q(I; L^r) \cap L^\infty(I; L^2), \ \forall A \in \{Id, J, H\} \right\} .$$

We first prove that there exists $0 < T < \frac{\pi}{4\omega_2}$ such that the set

$$\begin{aligned} X_T := \Big\{ u \in Y_r(]-T, T[); \ & \|A(t)u\|_{L^2} \leq 2R , \ \ \forall |t| < T, \ A \in \{Id, J, H\}, \\ & \|A(t)u\|_{L^q(]-T,T[;L^r)} \leq 2c_q \|\mathtt{w}\mathbb{1}_{]-\frac{\pi}{2\omega_2}, \frac{\pi}{2\omega_2}[}\|_{L^1_w}^{1/q} R, \ \forall A \in \{Id, J, H\} \Big\} \end{aligned}$$

is stable under the map F, defined in the previous section (right hand side in Duhamel's formula). The constant c_q is the one that appears in Lemma 2.1. We then prove that up to choosing T even smaller, F is a contraction on X_T. The natural norm on Y_r is

$$\|u\|_{Y_r} := \sum_{A \in \{Id, J, H\}} \left(\|A(t)u\|_{L^\infty(]-T,T[;L^2)} + \|A(t)u\|_{L^q(]-T,T[;L^r)} \right) .$$

For any pair (a, b), we use the notation $\|f\|_{L^a_T(L^b)} = \|f\|_{L^a(]-T,T[;L^b)}$. Let $u \in X_T$, and $A \in \{Id, J, H\}$. We already noticed that

$$\|A(t)F(u)\|_{L^\infty_T(L^2)} \leq R + \widetilde{C} \, \|u\|^{2\sigma}_{L^k_T(L^r)} \, \|A(t)u\|_{L^q_T(L^r)} ,$$

where $\widetilde{C}$ does not depend on ω_1 or ω_2. From Lemma 3.1, we have for $T < \frac{\pi}{4\omega_2}$, $\|u\|_{L^k_T(L^r)} \lesssim RT^{1/k}$. It follows,

$$\|A(t)F(u)\|_{L^\infty_T(L^2)} \leq R + CR^{2\sigma+1}T^{2\sigma/k} \ . \tag{4.1}$$

Use Lemma 2.1 to obtain

$$\begin{aligned}\|A(t)F(u)\|_{L^q_T(L^r)} &\leq c_q \left\|\mathtt{w}\mathbb{1}_{]-\frac{\pi}{2\omega_2},\frac{\pi}{2\omega_2}[}\right\|_{L^1_w}^{1/q} R + C\,\|u\|^{2\sigma}_{L^k_T(L^r)}\,\|A(t)u\|_{L^q_T(L^r)}\\ &\leq c_q \left\|\mathtt{w}\mathbb{1}_{]-\frac{\pi}{2\omega_2},\frac{\pi}{2\omega_2}[}\right\|_{L^1_w}^{1/q} R + CR^{2\sigma+1}T^{2\sigma/k} \ .\end{aligned}$$

It is now clear that if T is sufficiently small, then X_T is stable under F.

To complete the proof of the proposition, it is enough to prove contraction for small T in the weaker metric $L^q(]-T,T[;L^r)$. We have

$$\begin{aligned}\big\|F(u_2)-F(u_1)\big\|_{L^q_T(L^r)} &\leq C\,\big\|\big(|u_2|^{2\sigma}u_2 - |u_1|^{2\sigma}u_1\big)\big\|_{L^{q'}_T(L^{r'})}\\ &\leq C\left(\|u_1\|^{2\sigma}_{L^k_T(L^r)} + \|u_2\|^{2\sigma}_{L^k_T(L^r)}\right)\|u_2-u_1\|_{L^q_T(L^r)}.\end{aligned}$$

As above, we have the estimate $\|u_j\|^{2\sigma}_{L^k_T(L^r)} \leq CR^{2\sigma}T^{2\sigma/k}$, $j=1,2$. Therefore, contraction follows for T sufficiently small. The only requirements we make are $T < \frac{\pi}{4\omega_2}$ and $T \leq \eta = \eta(n,\sigma,|\lambda|,\|u_0\|_\Sigma)$. Therefore, we can find $\delta > 0$ independent of ω_1 and ω_2 such that $T \geq \frac{2\delta}{1+\omega_2}$, and the proposition follows with $T = 2t_0$. □

Remark 4.3. As pointed out by the referee, the approach of Proposition 4.1 could be pursued in order to complete the proof of Theorem 1.3 in just one step. Indeed, it is enough to prove that $\|u\|^{2\sigma}_{L^k(\mathbb{R};L^r)}$ can be made small under the assumptions of Theorem 1.3. We saw that choosing t_0 sufficiently small, $\|u\|^{2\sigma}_{L^k_{t_0}L^r}$ is small. Then using Lemma 3.1, the exponential decay shows that $\|u\|^{2\sigma}_{L^k(\{|t|>t_0\};L^r)}$ can be made small for ω_1 large enough (see the proof below). However, we chose to keep our initial presentation, for we believe it gives more information on the interaction between linear (due to the potential) and nonlinear effects.

Now the idea is that at time $t = t_0$, the L^p norms of u are small for $\omega_1 \gg 1+\omega_2$ ($p \neq 2$), from Lemma 3.1. In that case, the nonlinearity becomes negligible. Define the "approximate" solution v by

$$i\partial_t v + \frac{1}{2}\Delta v = \frac{1}{2}\left(-\omega_1^2x_1^2 + \omega_2^2x_2^2\right)v \quad ; \quad v\big|_{t=t_0} = u\big|_{t=t_0}\,, \tag{4.2}$$

where t_0 stems from Proposition 4.1. We also define the error $w = u - v$. Since v is defined globally, u exists globally in time if and only if w does.

We prove Theorem 1.3 for positive times. The proof for negative times is similar. The remainder w solves

$$i\partial_t w + \frac{1}{2}\Delta w = \frac{1}{2}\left(-\omega_1^2x_1^2 + \omega_2^2x_2^2\right)w + \lambda|u|^{2\sigma}u \quad ; \quad w\big|_{t=t_0} = 0\,. \tag{4.3}$$

We note that since the operators J and H commute with the linear part of (1.4), the conservation of mass for Schrödinger equations yields a constant C_1 independent of $\omega_1,\omega_2 > 0$ such that

$$\sum_{A\in\{Id,J,K\}} \|A(t)v\|_{L^2} \leq C_1\,, \tag{4.4}$$

uniformly in $t \in \mathbb{R}$. In view of Theorem 1.3, we may assume $\omega_1 \geq \omega_2$, so that Lemma 2.1 and (2.11) yield global in time Strichartz estimates, independent of ω_1 and ω_2. From Proposition 4.1, there exists $t_1 > 0$ such that w satisfies the same estimates as v on $[t_0, t_0 + t_1]$. So long as this holds, Strichartz estimates, together with Hölder's inequality, yield:

$$\begin{aligned}\|w\|_{L^q(I;L^r)} &\lesssim \left\||u|^{2\sigma}u\right\|_{L^{q'}(I;L^{r'})} \lesssim \|u\|^{2\sigma}_{L^k(I;L^r)}\|u\|_{L^q(I;L^r)} \\ &\lesssim \left(\|v\|^{2\sigma}_{L^k(I;L^r)} + \|w\|^{2\sigma}_{L^k(I;L^r)}\right)\left(\|v\|_{L^q(I;L^r)} + \|w\|_{L^q(I;L^r)}\right),\end{aligned} \tag{4.5}$$

for $I = [t_0, t]$. We have, from Lemma 3.1 and (4.4),

$$\|v(t)\|_{L^r} \lesssim \left(\frac{1+\omega_2}{\cosh(\omega_1 t)}\right)^{\delta(s)/n} = \left(\frac{1+\omega_2}{\cosh(\omega_1 t)}\right)^{\sigma/(2\sigma+2)},$$

therefore

$$\begin{aligned}\|v\|^k_{L^k(I;L^r)} &\lesssim \int_{t_0}^t \left(\frac{1+\omega_2}{\cosh(\omega_1\tau)}\right)^{2\sigma^2/(2-(n-2)\sigma)} d\tau \\ &\lesssim (1+\omega_2)^{2\sigma^2/(2-(n-2)\sigma)} \int_{t_0}^t \exp\left(-\frac{2\omega_1\sigma^2\tau}{2-(n-2)\sigma}\right) d\tau \\ &\lesssim \frac{(1+\omega_2)^{2\sigma^2/(2-(n-2)\sigma)}}{\omega_1} \exp\left(-\frac{2\omega_1\sigma^2 t_0}{2-(n-2)\sigma}\right) \\ &\lesssim \frac{(1+\omega_2)^{2\sigma^2/(2-(n-2)\sigma)}}{\omega_1} \exp\left(-\frac{2\omega_1\sigma^2\delta}{(2-(n-2))(1+\omega_2)}\right),\end{aligned} \tag{4.6}$$

where we used Proposition 4.1. So long as w satisfies (4.4), the L^qL^r norm of w on the right hand side of (4.5) can be absorbed, provided that (1.5) is satisfied for some sufficiently large Λ. Note that in general, we cannot do without the last term of (1.5), since for $n \geq 2$ and $\sigma \geq \frac{2}{n}$, $\frac{2\sigma^2}{2-(n-2)\sigma}$ may be smaller or larger than 1.

We therefore get an estimate for $\|w\|_{L^q(I;L^r)}$. Using Strichartz estimates again,

$$\begin{aligned}\|w\|_{L^\infty(I;L^2)} &\lesssim \left\||u|^{2\sigma}u\right\|_{L^{q'}(I;L^{r'})} \\ &\lesssim \|u\|^{2\sigma}_{L^k(I;L^r)}\|u\|_{L^q(I;L^r)} \\ &\lesssim \left(\|v\|^{2\sigma}_{L^k(I;L^r)} + \|w\|^{2\sigma}_{L^k(I;L^r)}\right)\left(\|v\|_{L^q(I;L^r)} + \|w\|_{L^q(I;L^r)}\right) \\ &\lesssim \left(\frac{(1+\omega_2)^{2\sigma^2/(2-(n-2)\sigma)}}{\omega_1}\right)^{2\sigma/k} e^{-\gamma\frac{\omega_1}{1+\omega_2}},\end{aligned}$$

for some positive γ. Applying the operator A, we get similar estimates. We conclude by a continuity argument that w satisfies (4.4) for all $t \geq t_0$ provided that Λ is sufficiently large in (1.5), and u is global, with:

$$Au \in L^\infty(\mathbb{R}; L^2) \cap L^q(\mathbb{R}; L^r), \quad \forall A \in \{Id, J, H\}.$$

Scattering follows easily. Indeed, we have from Duhamel's principle:

$$\|U_V(-\tau)u(\tau) - U_V(-t)u(t)\|_\Sigma = \left\|\lambda\int_t^\tau U_V(t-s)|u|^{2\sigma}u(s)ds\right\|_\Sigma.$$

From Strichartz and Hölder inequalities, we have:

$$\begin{aligned}\left\|\int_t^\tau U_V(t-s)|u|^{2\sigma}u(s)ds\right\|_\Sigma &\lesssim \|u\|^{2\sigma}_{L^k([t,\tau];L^r)}\times\\ &\times\sum_{A\in\{Id,J,H\}}\|Au\|_{L^q([t,\tau];L^r)}\to 0 \quad \text{as } t,\tau\to+\infty\,,\end{aligned}$$

since the L^kL^r norm goes to 0 exponentially, and the L^qL^r norms are bounded.

5. Discussion

For the sake of simplicity, we discuss our results in the case

$$V(x)=\frac{1}{2}\left(-\omega_1^2x_1^2+\omega_2^2x_2^2\right)\,. \tag{5.1}$$

We proved global existence and scattering when $\sigma\geq\frac{2}{n}$, and either $\|u_0\|_\Sigma$ is small, or (1.5) is satisfied. One expects that global existence (and scattering) should hold in Σ when the nonlinearity is defocusing, that is when $\lambda>0$. In the absence of *a priori* estimates (E_V is not signed), we cannot prove it. When $\sigma<\frac{2}{n}$ and $u_0\in\Sigma$, we proved global existence, but not scattering. Of course, it holds when $\|u_0\|_\Sigma$ is small, or (1.5) is satisfied. What happens in the general case is not clear. On the other hand, proving the existence of wave operators with asymptotic states in Σ for any $\sigma>0$ and any $\omega_1>0$, $\omega_2\geq 0$ is straightforward, thanks to the exponential decay (see [5] for a proof which can be easily adapted). We have more generally:

Proposition 5.1. *Let $\lambda\in\mathbb{R}$, and $\sigma>0$, with $\sigma<\frac{2}{n-2}$ if $n\geq 3$. Suppose that V is of the form* (1.3). *Then for every $u_-\in\Sigma$, there exist $T\in\mathbb{R}$ (finite) and a unique $u\in C(]-\infty,T];\Sigma)\cap L^q\left(]-\infty,T];L^{2\sigma+2}(\mathbb{R}^n)\right)$, where $q=\frac{4\sigma+4}{n\sigma}$, solution to* (1.1) *such that*

$$\|U_V(-t)u(t)-u_-\|_\Sigma\underset{t\to-\infty}{\longrightarrow}0\,.$$

The same holds for positive times and $u_+\in\Sigma$.

The condition (1.5) is nonlinear with respect to ω_2, which may seem surprising. On the other hand, notice that the constant in front of $(1+\omega_2)\ln(1+\omega_2)$ is universal, unlike Λ. This nonlinear factor vanishes when $\omega_2=0$; it arises in the estimate (4.6), and seems to be unavoidable. It is due to the concentrations which are caused by the harmonic potential (it is a linear phenomenon), and at time $t=\frac{\pi}{2\omega_2}$, this effect is not yet balanced by the repulsive harmonic potential in the first direction.

We do not know a criterion for global existence (or, equivalently, finite time "blow up") other than boundedness in Σ. One would expect the unboundedness of $\|\nabla_x u(t)\|_{L^2}$ to be the only obstruction to global existence. It turns out that this is true in the case of the *isotropic* repulsive harmonic potential studied in [5]. Proving this was not straightforward though, and used a particular evolution law, which does not seem to be available when the potential is not isotropic.

When there is no potential, or when the potential is non-negative (slightly more general potential are allowed, see [8]), a sufficient condition for finite time blow-up is provided by the virial theorem, following the ideas of [16]. It relies in the existence of a relatively simple evolution law for $\|xu(t)\|^2_{L^2}$. Sufficient conditions for finite time blow-up in the case of the isotropic harmonic potential, or the isotropic

repulsive harmonic potential were obtained in a similar way in [3, 5], using ordinary differential equations techniques. In the case of the potential (5.1), it seems that no such simplification is here to help.

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References

[1] N. Burq, P. Gérard, and N. Tzvetkov, *On nonlinear Schrödinger equations in exterior domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), no. 3, 295–318.
[2] N. Burq and F. Planchon, *Smoothing and dispersive estimates for 1D Schrödinger equations with BV coefficients and applications*, preprint, `math.AP/0409379`, 2004.
[3] R. Carles, *Remarks on nonlinear Schrödinger equations with harmonic potential*, Ann. Henri Poincaré **3** (2002), no. 4, 757–772.
[4] R. Carles, *Semi-classical Schrödinger equations with harmonic potential and nonlinear perturbation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), no. 3, 501–542.
[5] R. Carles, *Nonlinear Schrödinger equations with repulsive harmonic potential and applications*, SIAM J. Math. Anal. **35** (2003), no. 4, 823–843.
[6] R. Carles and L. Miller, *Semiclassical Nonlinear Schrödinger equations with potential and focusing initial data*, Osaka J. Math. **41** (2004), no. 3, 693–725.
[7] R. Carles and Y. Nakamura, *Nonlinear Schrödinger equations with Stark potential*, Hokkaido Math. J. **33** (2004), no. 3, 719–729.
[8] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.
[9] T. Cazenave and F. Weissler, *Some remarks on the nonlinear Schrödinger equation in the critical case*, Lect. Notes in Math., vol. 1394, Springer-Verlag, Berlin, 1989, pp. 18–29.
[10] W. Craig, T. Kappeler, and W. Strauss, *Microlocal dispersive smoothing for the Schrödinger equation*, Comm. Pure Appl. Math. **48** (1995), no. 8, 769–860.
[11] S.-I. Doi, *Smoothing effects for Schrödinger evolution equation and global behavior of geodesic flow*, Math. Ann. **318** (2000), 355–389.
[12] N. Dunford and J. T. Schwartz, *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*, With the assistance of William G. Bade and Robert G. Bartle, Interscience Publishers John Wiley & Sons New York-London, 1963.
[13] R. P. Feynman and A. R. Hibbs, *Quantum mechanics and path integrals (International Series in Pure and Applied Physics)*, Maidenhead, Berksh.: McGraw-Hill Publishing Company, Ltd., 365 p., 1965.
[14] D. Fujiwara, *A construction of the fundamental solution for the Schrödinger equation*, J. Analyse Math. **35** (1979), 41–96.
[15] D. Fujiwara, *Remarks on the convergence of the Feynman path integrals*, Duke Math. J. **47** (1980), no. 3, 559–600.
[16] R. T. Glassey, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), 1794–1797.
[17] L. Hörmander, *Symplectic classification of quadratic forms, and general Mehler formulas*, Math. Z. **219** (1995), no. 3, 413–449.
[18] L. Kapitanski and I. Rodnianski, *Regulated smoothing for Schrödinger evolution*, Internat. Math. Res. Notices (1996), no. 2, 41–54.
[19] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), no. 5, 955–980.
[20] E. H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.
[21] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.
[22] G. Staffilani and D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Comm. Partial Differential Equations **27** (2002), no. 7-8, 1337–1372.

[23] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N. J., 1970.

[24] Y. Tsutsumi, L^2*–solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. **30** (1987), no. 1, 115–125.

MAB, UMR CNRS 5466, Université Bordeaux 1, 351 cours de la Libération, 33 405 Talence cedex, France

E-mail address: Remi.Carles@math.u-bordeaux1.fr

URL: http://www.math.u-bordeaux1.fr/~carles